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Special Form Hankel Matrix Inverses $(n+1)\times(n+1), n \geq 3$ With 2×2 Block Matrices

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ABSTRACT

This study aims to determine the inverse of a special form Hankel matrix using a 2×2 *block matrix. In this study, some steps are carried out. The first step will be given a special form Hankel matrix which will then be blocked into a* 2×2 *block matrix. Next, determine the inverse of the invertible submatrix of the Hankel matrix so that the general form is obtained. The last step is seen from the inverse pattern of the two ways of blocking the Hankel matrix of the unique structure of order* 4×4 *to* 8×8 *so that the general shape of the inverse Hankel matrix of special form is obtained. The results obtained will be obtained in the general structure of the Hankel matrix inverse special form* $(n+1)\times(n+1), (n\geq 3)$ *using a* 2×2 *block matrix.*

Keyword: *Matrix block* 2×2*; Inverse; Hankel Matrix*

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1. INTRODUCTION

The matrix inverse can solve problems in a system of linear equations (Marzuki and Aryani 2019). Inverse is the inverse of a matrix (Howard and Rorres 2004). Some methods used to calculate the inverse of a matrix include the adjoin method, elementary row operation method, 2×2block matrix and so on. In 2017, there was research (Fran n.d.) discussing determinants and inverses of block matrices in general using a 2×2 block matrix. The general form of the 2×2block matrix is as follows:

$$P = \left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1(n-k)} & a_{1(n-(k-1))} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & v & \cdots & \vdots \\ a_{(m-k)1} & \cdots & a_{(m-k)(n-k)} & a_{(m-k)(n-(k-1))} & \cdots & a_{(m-k)n} \\ \hline a_{(m-(k-1))1} & \cdots & a_{(m-(k-1)(n-k))} & a_{(m-(k-1)(n-(k-1))))} & \cdots & a_{(m-(k-1))n} \\ \vdots & \cdots & \vdots & \ddots & \cdots & \vdots \\ a_{m1} & \cdots & a_{m(n-k)} & a_{m(n-(k-1))} & \cdots & a_{mn} \end{array}\right] \quad (1.1)$$

The results obtained to determine the determinant of the matrix P by supposing the submatrices A and D or B and C are square matrices. If A and D are square matrices then $\det(P) = \det(A)\det(D - CA^{-1}B)$ if $\det(A) \neq 0$ or $\det(P) = \det(D)\det(A - BD^{-1}C)$ if $\det(D) \neq 0$. If B and C are square matrices then we get $\det(P) = (-1)^{\left(\frac{1}{2}(n^2-2n+p^2+q^2)\right)}\det(B)\det(C - AB^{-1}D)$ or using $\det(P) = (-1)^{\left(\frac{1}{2}(n^2-2n+p^2+q^2)\right)}\det(C)\det(B - DC^{-1}A)$. Meanwhile, to determine the inverse of matrix P, it is determined by supposing the submatrix, B, C or D with a determinant not equal to zero and applying the Schur complement method. Research (Rahma, Anggelina, and Rahmawati 2019) which discusses the general form of the inverse of the 2×2 block matrix in the application of the matrix $FLDcirc_r$ with a special form with $P_n = FLDcirc_r(0,0,a,0,\ldots,0)$.

The results of the discussion are:

$$(P_n)^{-1} = \begin{bmatrix} a^{-1} & 0 & a & 0 & 0 & \ldots & 0 & 0 & 0 & (ra)^{-1} & (ra)^{-1} \\ a^{-1} & 0 & 0 & a & 0 & \ldots & 0 & 0 & 0 & 0 & (ra)^{-1} \\ a^{-1} & 0 & 0 & 0 & a & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a^{-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & a^{-1} & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & a^{-1} & 0 & 0 \end{bmatrix}$$

Furthermore, research (Irawan 2022) which discusses determining the general form of the inverse of the Toeplitz matrix with a special form using the 2×2 block matrix method. The results of the discussion are:

$$T_n^{-1} = \begin{bmatrix} -\frac{n-2}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \cdots & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} \\ \frac{1}{(n-1)a} & -\frac{n-2}{(n-1)a} & \frac{1}{(n-1)a} & \cdots & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} \\ \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & -\frac{n-2}{(n-1)a} & \cdots & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \cdots & -\frac{n-2}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} \\ \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \cdots & \frac{1}{(n-1)a} & -\frac{n-2}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} \\ \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \cdots & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & -\frac{n-2}{(n-1)a} & \frac{1}{(n-1)a} \\ \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \cdots & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & \frac{1}{(n-1)a} & -\frac{n-2}{(n-1)a} \end{bmatrix}$$

Based on the description above and related studies, the authors are interested in conducting research that discusses the special form Hankel matrix, namely determining the inverse of the Hankel matrix with the research title "Hankel Matrix Inverse Special Form $(n+1) \times (n+1)$, $(n \geq 3)$ with 2×2Block Matrix" with the following special form:

$$H_{n+1} = \begin{bmatrix} a & b & 0 & 0 & \dots & 0 & 0 & a & b \\ b & 0 & 0 & 0 & \dots & 0 & a & b & 0 \\ 0 & 0 & 0 & 0 & \dots & a & b & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & b & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & a & b & \dots & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & \dots & 0 & 0 & 0 & 0 \\ a & b & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix} \text{with } a, b \in \mathbb{R} \; for \; a, b \neq 0 \quad (1.2)$$

2. RESEARCH METHODS

This research is a literature study research where there are research steps first, given a Hankel matrix in Equation (1.2). Second, Blocking the Hankel matrix into a 2×2 block matrix starting from order4×4 to 8×8 which has each submatrix by initializing with A, B, C and D. Third, Applying Schur complement to the invertible submatrix of the Hankel matrix of order 4×4 to 8×8. Fourth, Determine the inverse of the Hankel matrix of 4×4 to 8×8 based on Theorem 2.3 parts (iii) and (iv). Fifth, Estimating the general form of the invertible submatrix and the inverse of the Hankel matrix H_{n+1}in general by observing the pattern. Sixth, Proving the general form of the inverse of the invertible submatrix of Hankel matrix $H_{n+1}$by using the inverse rule. Finally, Proving the general form of the general inverse of the Hankel matrix $H_{n+1}$by proving $(H_{n+1})(H_{n+1})^{-1} = (H_{n+1})^{-1}(H_{n+1}) = I$.

The following is given the theoretical basis or materials needed in the discussion.

Definition 1 (Howard and Rorres 2004) A matrix that is blocked so that it becomes several matrix parts with a smaller size than the previous matrix and contains horizontal and vertical lines between the rows and columns of the matrix. Then, the small-sized matrix is called a submatrix.

Definition 2 (Fran n.d.) A 2×2 block matrix is a square matrix that is blocked or partitioned into two rows and two columns of submatrices. The general form of a 2×2block matrix is as follows:

Suppose P is a matrix of size $m \times n$

$$P = \begin{bmatrix} a_{11} & \cdots & a_{1(n-k)} & a_{1(n-(k-1))} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & v & \cdots & \vdots \\ a_{(m-k)1} & \cdots & a_{(m-k)(n-k)} & a_{(m-k)(n-(k-1))} & \cdots & a_{(m-k)n} \\ a_{(m-(k-1))1} & \cdots & a_{(m-(k-1)(n-k))} & a_{(m-(k-1)(n-(k-1)))} & \cdots & a_{(m-(k-1))n} \\ \vdots & \cdots & \vdots & \ddots & \cdots & \vdots \\ a_{m1} & \cdots & a_{m(n-k)} & a_{m(n-(k-1))} & \cdots & a_{mn} \end{bmatrix}$$

Next, draw horizontal and vertical lines so that the matrix becomes as follows:

$$P = \left[\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1(n-k)} & a_{1(n-(k-1))} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & v & \cdots & \vdots \\ a_{(m-k)1} & \cdots & a_{(m-k)(n-k)} & a_{(m-k)(n-(k-1))} & \cdots & a_{(m-k)n} \\ \hline a_{(m-(k-1))1} & \cdots & a_{(m-(k-1)(n-k))} & a_{(m-(k-1)(n-(k-1)))} & \cdots & a_{(m-(k-1))n} \\ \vdots & \cdots & \vdots & \ddots & \cdots & \vdots \\ a_{m1} & \cdots & a_{m(n-k)} & a_{m(n-(k-1))} & \cdots & a_{mn} \end{array}\right]$$

Then it is generalized:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1(n-1)} \\ \vdots & \ddots & \vdots \\ a_{(m-k)1} & \cdots & a_{(m-k)(n-k)} \end{bmatrix}, B = \begin{bmatrix} a_{1(n-(k-1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m(n-(k-1)} & \cdots & a_{mn} \end{bmatrix}$$

$$C = \begin{bmatrix} a_{(m-(k-1)1} & \cdots & a_{(m-(k-1)(n-k)} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{m(n-k)} \end{bmatrix},$$

$$D = \begin{bmatrix} a_{(m-(k-1)(n-(k-1)} & \cdots & a_{(m(k-1))n} \\ \vdots & \ddots & \vdots \\ a_{m(n-k-1)} & \cdots & a_{mn} \end{bmatrix}$$

Then: $P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

Definition 3 (Yang and Dong 2018) The nth Hankel matrix of order $a = (a_n)$ for $n \geq 0$ is an $(n+1) \times (n+1)$ matrix whose (i,j) entries are a_a_{i+j}

Hankel matrix is a matrix whose every element along the diagonal $i + j$ is constant. This means that every slope of the matrix diagonal from left to right is constant (Aqilah 2020). The general form of the Hankel matrix is a square matrix A matrix of size $(n+1) \times (n+1)$ is as follows:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_n \\ a_1 & a_2 & a_3 & \cdots & a_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & a_{n+2} & \cdots & a_{2n} \end{bmatrix}$$

Definition 4 (Redivo-Zaglia 2004) One method for analyzing matrices that makes extensive use of matrix inequalities is the schur complement. In its use, large matrices that have been blocked before often use schur complements.

Given a matrix:

$$\underset{(j+k)\times(l+m)}{P} = \begin{pmatrix} \underset{j\times l}{A} & \underset{j\times m}{B} \\ \underset{k\times l}{C} & \underset{k\times m}{D} \end{pmatrix}$$

1. If A is an invertible matrix, the schur complement of A is $D - CA^{-1}B$.
2. If D is an invertible matrix, the schur complement of D is $A - BD^{-1}C$.
3. If B is an invertible matrix, the schur complement of B is $C - DB^{-1}A$.
4. If C is an invertible matrix, the schur complement of C is $B - AC^{-1}D$.

Definition 5 (Lu and Shiou 2002) A non-singular matrix P with P^{-1} can be blocked into a 2×2 block matrix by blocking the matrix as follows:

$P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ dan $P^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$

To multiply a matrix P by P^{-1} and P^{-1} by P cannot be done with arbitrary size or order (Irawan 2022). Suppose A, B, C, and D have sizes $j \times l$, $j \times m$, $k \times l$, and $k \times m$ for $j + k = l + m$. Furthermore E, F, G, and H must have sizes $l \times j$, $l \times k$, $m \times j$, and $m \times k$. That is P^{-1} is in the transpose block of P.

Suppose a block of A, B, C, dan D is an unsingular square matrix. So as to avoid the matrix being generalized the possibility of having three blocks viz:

1. Diagonal block of squares viz: $j = l$ and $k = m$..
2. Square block diagonal: $j = m$ and $k = l$.
3. All square blocks: $j = m = l = m$.

Theorem 1 (Lu and Shiou 2002) If P is a square matrix, then:

i. For the matrix $P = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ will have an inverse if and only if A and D have inverses, then $P^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & D^{-1} \end{bmatrix}$.

ii. For the matrix $P = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ will have an inverse if and only if B and C have inverses, then $P^{-1} = \begin{bmatrix} 0 & B^{-1} \\ C^{-1} & 0 \end{bmatrix}$.

Theorem 2 (Lu and Shiou 2002) If P is a square matrix, then:

i. For matrix $P = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$, it will have an inverse if and only if A and D have inverses, so $P^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$.

ii. For a matrix $P = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$, it will have an inverse if and only A and D punya invers, so $P^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix}$.

Theorem 3 (Lu and Shiou 2002) Suppose P is a square matrix, then:

i. Suppose the submatrix A of the matrix P in Equation (1.1) is nonsingular. The matrix in Equation (2.1) has an inverse if and only if the schur complement of A has an inverse and (D-CA^(-1) B) also has an inverse, then we get:

$$P^{-1} = \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}$$

ii. Suppose the submatrix D of the matrix P in Equation (1.1) is nonsingular. The matrix in Equation (2.1) has an inverse if and only if the schur complement of D has an inverse and (D-BD^(-1)C) also has an inverse, then we get:

$$P^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}$$

iii. Suppose the submatrix B of the matrix P in Equation (2.1) is nonsingular. The matrix in Equation (2.1) has an inverse if and only if the schur complement of B has an inverse and (C-DB^(-1) A) also has an inverse, then we get:

$$P^{-1} = \begin{bmatrix} -(C - DB^{-1}A)^{-1}DB^{-1} & (C - DB^{-1}A)^{-1} \\ B^{-1} + B^{-1}A(C - DB^{-1}A)^{-1}DB^{-1} & -B^{-1}A(C - DB^{-1}A)^{-1} \end{bmatrix}$$

iv. Suppose the submatrix C of the matrix P in Equation (1.1) is nonsingular. The matrix in Equation (2.1) has an inverse if and only if the schur complement of C has an inverse and (B-AC^(-1)D) also has an inverse, then we get:

$$P^{-1} = \begin{bmatrix} -C^{-1}D(B - AC^{-1}D)^{-1} & C^{-1} + C^{-1}D(B - AC^{-1}D)^{-1}AC^{-1} \\ (B - AC^{-1}D)^{-1} & -(B - AC^{-1}D)^{-1}AC^{-1} \end{bmatrix}$$

Theorem 4 (Lu and Shiou 2002) If P is a square matrix, then:

i. For matrix $P = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}$, it has an inverse if and only if B and C have inverses such that $P^{-1} = \begin{bmatrix} -C^{-1}DB^{-1} & C^{-1} \\ B^{-1} & 0 \end{bmatrix}$.

ii. For matrix $P = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$, it has an inverse if and only if B dan C have inverses such that $P^{-1} = \begin{bmatrix} 0 & -C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}$

3. RESULTS AND DISCUSSION

After following the steps in the research method described above, 2 ways of invertible submatrices are obtained, from these 2 ways, only 1 way will be written starting from the inversible submatrix until the general form of the Hankel Matrix Inverse Special Form $(n + 1) \times (n + 1)$, $(n \geq 3)$ is obtained as follows:

Way 1.

1. For H_4 then

$$B^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{b} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2-b^2}{b^3} \end{bmatrix} \text{ and } C^{-1} = \begin{bmatrix} \frac{1}{b} \end{bmatrix}$$

2. For H_5 then

$$B^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & \frac{-(a^3+b^3)}{b^4} \end{bmatrix} \text{ and } C^{-1} = \begin{bmatrix} \frac{1}{b} \end{bmatrix}$$

3. For H_6 then

$$B^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4-b^4}{b^5} \end{bmatrix} \text{ and } C^{-1} = \left[\frac{1}{b}\right]$$

4. For H_7 then

$$B^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} \\ 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} & \frac{-(a^5+b^5)}{b^6} \end{bmatrix} \text{ and } C^{-1} = \left[\frac{1}{b}\right]$$

5. For H_8 then

$$B^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} \\ 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} \\ 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} & -\frac{a^5}{b^6} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} & -\frac{a^5}{b^6} & \frac{a^6-b^6)}{b^7} \end{bmatrix} and\ C^{-1} = \left[\frac{1}{b}\right]$$

Then in the same way, the inverse of the special form Hankel matrix of order 4×4 to 8×8 is determined as follows.

1. For H_4 then

$$(H_4)^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2-b^2}{b^3} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2-b^2}{b^3} & -\frac{(a^2-b^2)a}{b^4} \end{bmatrix}$$

2. For H_5 then

$$(H_5)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{(a^3+b^3)}{b^4} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{(a^3+b^3)}{b^4} & \frac{(a^3+b^3)a}{b^5} \end{bmatrix}$$

3. For H_6 then

$$(H_6)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} \\ 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4-b^4}{b^5} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4-b^4}{b^5} & \frac{-(a^4-b^4)a}{b^6} \end{bmatrix}$$

4. For H_7 then

$$(H_7)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} \\ 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} \\ 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} & \frac{-(a^5+b^5)}{b^6} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} & \frac{-(a^5+b^5)}{b^6} & \frac{(a^5+b^5)a}{b^7} \end{bmatrix}$$

5. For H_8 then

$$(H_8)^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} \\ 0 & 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} \\ 0 & 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} \\ 0 & 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} & -\frac{a^5}{b^6} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} & -\frac{a^5}{b^6} & \frac{(a^6-b^6)}{b^7} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & -\frac{a^3}{b^4} & \frac{a^4}{b^5} & -\frac{a^5}{b^6} & \frac{(a^6-b^6)}{b^7} & \frac{-(a^6+b^6)a}{b^8} \end{bmatrix}$$

Next, the general form of the inverse of the special form Hankel submatrix $(n+1)\times(n+1)$ will be estimated, namely:
Way 1.

$$B^{-1}=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & 0 & 0 & \dots & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \frac{1}{b} & \dots & \frac{(-a)^{n-5}}{b^{n-4}} & \frac{(-a)^{n-4}}{b^{n-3}} & \frac{(-a)^{n-3}}{b^{n-2}} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \dots & \frac{(-a)^{n-4}}{b^{n-3}} & \frac{(-a)^{n-3}}{b^{n-2}} & \frac{(-a)^{n-2}}{b^{n-1}} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & \dots & \frac{(-a)^{n-3}}{b^{n-2}} & \frac{(-a)^{n-2}}{b^{n-1}} & \frac{(-a)^{n-1}-b^{n-1}}{b^n} \end{bmatrix} \text{ and } C^{-1}=\begin{bmatrix}\frac{1}{b}\end{bmatrix}$$

From the conjecture, the general form of the invertible submatrix inverse of the special form Hankel matrix $(n+1)\times(n+1)$ will be addressed in Theorem 5 and Theorem 6 as follows:

Theorem 5. Given a special form of Hankel matrix in Equation (1.2) using Method 1, an invertible submatrix will be obtained, namely:

$$B=\begin{bmatrix} b & 0 & 0 & \dots & 0 & a & b \\ 0 & 0 & 0 & \dots & a & b & 0 \\ 0 & 0 & 0 & \dots & b & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & a & b & \dots & 0 & 0 & 0 \\ a & b & 0 & \dots & 0 & 0 & 0 \\ b & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} \text{ and } C=[b]$$

Hence, the inverse of the invertible submatrices B and C is:

$$B^{-1}=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & 0 & 0 & \dots & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \frac{1}{b} & \dots & \frac{(-a)^{n-5}}{b^{n-4}} & \frac{(-a)^{n-4}}{b^{n-3}} & \frac{(-a)^{n-3}}{b^{n-2}} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \dots & \frac{(-a)^{n-4}}{b^{n-3}} & \frac{(-a)^{n-3}}{b^{n-2}} & \frac{(-a)^{n-2}}{b^{n-1}} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & \dots & \frac{(-a)^{n-3}}{b^{n-2}} & \frac{(-a)^{n-2}}{b^{n-1}} & \frac{(-a)^{n-1}-b^{n-1}}{b^n} \end{bmatrix} \text{ and } C^{-1}=\begin{bmatrix}\frac{1}{b}\end{bmatrix}$$

Proof:

Based on the Definition, we prove Theorem 5 by showing that there exists an identity matrix I such that $BB^{-1}=B^{-1}B=I$.

For the left inverse: $BB^{-1}=I$

$$BB^{-1}=\begin{bmatrix} b & 0 & 0 & \cdots & 0 & a & b \\ 0 & 0 & 0 & \cdots & a & b & 0 \\ 0 & 0 & 0 & \cdots & b & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & a & b & \cdots & 0 & 0 & 0 \\ a & b & 0 & \cdots & 0 & 0 & 0 \\ b & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}_n \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{b} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{b} & -\frac{a}{b^2} \\ 0 & 0 & 0 & \cdots & \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \frac{1}{b} & \cdots & \frac{(-a)^{n-5}}{b^{n-4}} & \frac{(-a)^{n-4}}{b^{n-3}} & \frac{(-a)^{n-3}}{b^{n-2}} \\ 0 & \frac{1}{b} & -\frac{a}{b^2} & \cdots & \frac{(-a)^{n-4}}{b^{n-3}} & \frac{(-a)^{n-3}}{b^{n-2}} & \frac{(-a)^{n-2}}{b^{n-1}} \\ \frac{1}{b} & -\frac{a}{b^2} & \frac{a^2}{b^3} & \cdots & \frac{(-a)^{n-3}}{b^{n-2}} & \frac{(-a)^{n-2}}{b^{n-1}} & \frac{(-a)^{n-1}-b^{n-1}}{b^n} \end{bmatrix}_n$$

$BB^{-1}=I$ Proven.

4. CONCLUSIONS

Based on the discussion that has been done, it is concluded that: with 2×2 Block Matrix shows that the Hankel matrix of size (n+1)×(n+1), with elements that have a symmetrical pattern, can be analyzed using a 2×2 block matrix approach to facilitate the calculation of its inverse. In this article, it is explained that the block matrix structure allows the separation and completion of smaller sub-matrices, so that the inverse calculation process becomes more efficient and structured, especially for matrices with larger sizes (n ≥ 3). This technique provides a new view in solving the Hankel matrix inverse problem in a more practical and comprehensive manner. The general form of Hankel matrix inverse of special form (n+1)×(n+1), (n≥3) using 2×2 block matrix is obtained in Theorem 5.

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